Assignment 3: Langevin and Fokker–Planck equations.

What you will learn by solving this exercise: Fourier method for solving Fokker-Planck equation, method of characteristic, and Greens function method.

First Problem: Ornstein–Uhlenbeck process

Consider an **overdamped** Langevin particle in a harmonic potential $U(x) = \frac{1}{2}\lambda x^2$. The particle position $x(t)$ follows a Langevin equation

$$\dot{x}(t) = -\gamma\, U'(x) + \eta(t) \tag{1}$$

with a Gaussian white noise $\eta(t)$ with

$$\langle \eta(t) \rangle = 0 \qquad \text{and} \qquad \langle \eta(t)\eta(t') \rangle = 2D\, \delta(t - t') \tag{2}$$

The particle starts at $x(0) = x_0$ at time $t = 0$.

1. Write the solution for $x(t)$ as a function of $\eta(t)$. Then, averaging over noise realisations, find expression for mean $\langle x(t) \rangle$ and variance $\langle x(t)^2 \rangle - \langle x(t) \rangle^2$. What can you say about higher cumulants of $x(t)$?
2. Write the corresponding Fokker-Planck equation for probability $P_t(x|x_0)$.
3. **Solution by Fourier method.**
    1. Use Fourier transformation

$$P_t(x|x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx}\, \widehat{P}_t(k|x_0) \tag{3}$$

and write the equation for $\widehat{P}_t(k|x_0)$.

2. Considering the initial condition, solve for $\widehat{P}_t(k|x_0)$. [Hint: use method of characteristics].

3. Taking the inverse Fourier transformation, find $P_t(x|x_0)$. Write the solution as

$$P_t(x|x_0) = \sqrt{\frac{1}{2\pi\sigma^2(t)}} \exp\left[-\frac{(x-y(t))^2}{2\sigma^2(t)}\right] \tag{4}$$

Can you interpret what these quantities $y(t)$ and $\sigma^2(t)$ are? Can you intuitively justify this solution [Hint: use Gaussianity and argue reason behind this].

4. What is the stationary distribution ($t \to \infty$) ? Comparing with what you may expect from a Boltzmann-Gibbs distribution, find a relation between γ, D and $k_B T$.

4. **Solution by mapping to Schrodinger equation.**

1. Write the explicit transformation to map the Fokker-Planck equation for $P_t(x|x_0)$ to a Schrodinger equation. Write the corresponding Schrodinger equation.

2. What are the energy eigenstates and eigenfunctions for this particular Schrodinger equation? Use the standard solution for eigenfunctions of a quantum harmonic oscillator, in terms of Hermite polynomials $H_n(x)$, to formally express $P_t(x|x_0)$ as a summation over Eigen states. Express it in a form

$$P_t(x|x_0) = \mathcal{N} e^{-f(x)} \sum_{n\geq 0} e^{-E_n t}\, C_n\, H_n(ax)\, H_n(ax_0) \tag{5}$$

and give explicit expression for all the unknowns in the formula.

3. Use the following identity for Hermite polynomials $H_n(x)$ (Mehler's formula)

$$\sum_{n\geq 0} \frac{H_n(x)H_n(y)}{n!}\left(\frac{\omega}{2}\right)^n = \frac{1}{\sqrt{1-\omega^2}}\exp\left[\frac{2xy\omega-(x^2+y^2)\omega^2}{1-\omega^2}\right] \tag{6}$$

to show that results for $P_t(x|x_0)$ is same as what you derived by Fourier method

Second problem : Random acceleration process

A Random Accelaration Process (RAP) is an extension of the Brownian motion. It is defined by

$$\frac{d^2x(t)}{dt^2} = \eta(t) \tag{7}$$

with the Gaussian noise $\eta(t)$ defined in eq (2). Note that this is a non-Markovian process.

1. Express the equation as a joint equation for position $x(t)$ - velocity $v(t)$ variable. Note, this makes the joint evolution Markovian.
2. Assume an initial condition $x(0) = 0$ and $v(0) = 0$. From this coupled equation, find $\langle x(t)\rangle$, $\langle x(t)^2\rangle$, $\langle v(t)\rangle$, $\langle v(t)^2\rangle$, and $\langle x(t)v(t)\rangle$. Can you say anything about their higher cumulantes?
3. Write the Fokker-Planck equation for $P_t(x, v|0, 0)$.
4. Without explicitely solving the problem, could you argue if the solution be a Gaussian? Then, using the moments derived in problem 2, write the explicit formula of $P_t(x, v|0, 0)$.
5. Derive the same result by explitely solving the Fokker-Planck equation using Fourier method.

Remark: (not for you to solve). A well-known simple model of Active matter is Active-Ornstein-Uhlenbeck process. This is a combination of the above two problems, and defined by

$$\dot{x} = \alpha\, v; \qquad \dot{v} = \lambda v + \eta(t) \tag{8}$$

This problem can be fully solved again by Fourier method and also by Gaussianity argument [see: sec 4 in New J. Phys. **20**, (2018) 015001]

Third problem: particle in a circular trap.

Consider a Brownian particle on a ring with periodicity 2π. Corresponding Fokker-Planck equation

$$\frac{\partial P_t}{\partial t} = D \frac{\partial^2 P_t}{\partial x^2} \tag{9}$$

1. Using Fourier method show that the solution

$$P_t(x|x_0) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n \geq 1} e^{-tDn^2} \cos[n\,(x - x_0)]. \tag{10}$$

Note that because of periodicity, the Fourier modes are discrete.

2. Now consider that there is a constant circular driving field f, such that the Fokker-Planck equation becomes

$$\frac{\partial P_t}{\partial t} = D \frac{\partial^2 P_t}{\partial x^2} - f\,\frac{\partial P_t}{\partial x} \tag{11}$$

What is the solution $P_t(x|x_0)$ in this case? [Hint: there is a simple modification of the above answer.]

3. In addition to the force f consider a periodic potential $U(x) = \lambda \cos(x)$. Write the corresponding Fokker-Planck equation.

4. We learned in the class that the stationary state in this case is outside equilibrium, and there is an explicit formula for the stationary probability. However, the time dependant solution is difficult. One way to solve this is by Green's function method.
Show that, the solution can be formally written as

$$P_t(x|x_0) = G_t(x|x_0) - \lambda \int_0^t ds \int dx' G_{t-s}(x|x') \frac{\partial}{\partial x'} [\sin(x') P_s(x'|x_0)] \,, \quad (12)$$

where $G_t(x|x_0)$ is the Greens function and the solution for $\lambda = 0$ case in eq. (11).

5. Write recursively the solution upto second order in λ, for small λ. (Similar to Dyson series.)